Professional Services Committee

Subject Matter Competence of Teachers of Mathematics

Executive Summary: This agenda item continues the discussion related to the teaching of mathematics in California. The focus of this item is the process for determining the subject matter competence of teachers of K-12 mathematics.

Recommended Action: For information only

Presenter: Phyllis Jacobson, Ed.D., Administrator, Professional Services Division

Strategic Plan Goal: 1

Promote educational excellence through the preparation and certification of professional educators

♦ Sustain high quality standards for the preparation and performance of professional educators and for the accreditation of credential programs

April 2009

Subject Matter Competence of Teachers of Mathematics

I. Introduction

The Commission began a discussion at its October 2008 meeting related to the preparation of individuals who teach mathematics (http://www.ctc.ca.gov/commission/agendas/2008-10/2008-10-2D.pdf). At the November 2008 Commission meeting (http://www.ctc.ca.gov/commission/agendas/2008-11/2008-11-2G.pdf) staff presented a plan for addressing the issues related to the authorizations to teach mathematics in California's public schools. The December 2008 item focused on the Mathematics Specialist Credential. In January 2009, staff provided a detailed overview of the authorizations that allow an individual to teach mathematics. This agenda item furthers the discussion by providing information about what constitutes subject matter competence in mathematics and how the subject matter competence of teachers to teach K-12 mathematics is assessed.

This agenda item focuses specifically on the *mathematics content knowledge* expected of beginning teachers, and looks at how this content knowledge is assessed through subject matter preparation programs and subject matter examinations. This item does not address teacher candidates' level of pedagogical knowledge about how to teach mathematics content to K-12 students, nor does it focus on the pedagogical preparation received by teacher candidates within teacher preparation programs about how to teach mathematics content to K-12 students. The pedagogical preparation of teacher candidates to teach mathematics will be the topic for the next agenda item in this series of information items, to be presented at the June 2009 Commission meeting.

Background

In the January 2009 agenda item, a table was presented that outlined all the authorizations to teach mathematics and the subject matter knowledge required for each authorization. This agenda item focuses on the three full authorizations to teach K-12 mathematics, as shown in the table excerpt below.

Table 1: Authorizations to Teach Mathematics

	Credential Type	Authorized Assignments	Grade Levels/ Settings	Subject Matter Preparation Required
	Basic Teaching Credentials			
A	Multiple Subject Credential (Includes Interns)	Math content grades 12 and below	Math in Self-Contained Classrooms or Core Settings[1/]	Passage of CSET: Multiple Subjects. Examination subject matter requirements are aligned to the K-7 academic content standards in mathematics. NCLB Compliant.

Table 1: Authorizations to Teach Mathematics

	Credential Type	Authorized Assignments	Grade Levels/ Settings	Subject Matter Preparation Required
B	Single Subject Credential in Mathematics (Includes Interns)	All mathematics courses	Departmentalized Classrooms—all Grade Levels	Passage of CSET: Mathematics (3 Sections) or completion of an approved subject matter program. Examination or program subject matter requirements are aligned to the 8-12 grade academic content standards in mathematics. NCLB Compliant.

	Credential Type	Authorized Assignments	Grade Levels/ Settings	Subject Matter Preparation Required
C	Single Subject Credential in Foundational-Level Mathematics (Includes Interns)	General Math, Consumer Math, Algebra, Geometry Probability and Statistics	Departmentalized Classrooms—all Grade Levels	Passage of CSET Foundational Mathematics (2 Sections) or completion of an approved foundational-level mathematics subject matter program. Examination or program subject matter requirements are aligned to the Algebra, Geometry, and Probability and Statistics academic content standards in mathematics. NCLB Compliant.

[1/] *A Core setting is when two or more subjects are taught to the same group of students, in grades 5-8 –Algebra 1 may be included as one of those subjects.*

II. Candidate Subject Matter Competence in Mathematics

A. Definition of Subject Matter Competence in Mathematics

The mathematics content that forms the basis for determining subject matter competence is defined as a set of subject matter requirements (SMRs) with respect to mathematics that beginning teachers are expected to know. Appendix A to this agenda item provides the list of the Mathematics SMRs for multiple subject teacher candidates. Appendix B provides the list of the Mathematics SMRs for single subject teacher candidates. These SMRs form the basis for both the candidate competencies contained within the mathematics subject matter preparation program standards and the mathematics content from which CSET examination items are developed. The SMRS are aligned with the K-12 student academic content standards in Mathematics and also with the applicable state framework for Mathematics. Multiple subject candidates who pass the CSET: MS Examination, and single subject candidates who successfully complete a Commission-approved subject matter preparation program in mathematics or who pass the CSET: Mathematics

examination, are deemed to be subject matter competent in mathematics for the grade levels authorized by their teaching credential.

Overview of the Mathematics Subject Matter Requirements for Multiple Subject Candidates

The SMRs for multiple subject teachers are organized around four major content domains: (1) Number Sense, (2) Algebra and Functions, (3) Measurement and Geometry, and (4) Statistics, Data Analysis, and Probability. These are the same domains around which the student academic content standards in mathematics are organized for grades K-7, with the exception of the "Mathematical Reasoning" domain. Within each of the major content domains listed above, the range of content is further delineated. For example, within the Measurement and Geometry domain, the subdomains include Two- and Three-dimensional Geometric Objects; Representational Systems including Concrete Models, Drawings, and Coordinate Geometry; and Techniques, Tools and Formulas for Determining Measurements. Within each subdomain, there is a more detailed description of the content included in that subdomain. For example, within the Two- and Three-dimensional Geometric Objects subdomain, the specific content referenced includes triangles, quadrilaterals, and spheres; congruence, similarity or lack thereof, of two figures; symmetry, translations, rotations, and reflections; the Pythagorean theorem and its converse; and properties of parallel lines. These subdomains and their specific content also reflect the organization of the student academic content standards.

In addition to the content specified within the domains, there are also statements about the subject matter skills and abilities applicable to the Mathematics content domains. For example, multiple subject candidates are expected to be able, among other skills, to analyze complex problems for alternative solution strategies; evaluate the truth of mathematical statements; and explain their mathematical reasoning. It is in describing the subject matter skills and abilities that the SMRs address the student academic content standards relating to Mathematical Reasoning.

Overview of the Mathematics Subject Matter Requirements for Single Subject Candidates

The mathematics subject matter requirements for single subject candidates are organized into six major domains: (1) Algebra; (2) Geometry; (3) Number Theory; (4) Probability and Statistics; (5) Calculus; and (6) History of Mathematics. Domains 5 and 6 are not applicable to candidates for the Foundational Level Mathematics credential. The domains are similar to the domains indicated in the grades 8-12 student academic content standards for individual mathematics-related subjects. These major content domains are delineated further by more specific content. For example, in the Geometry domain, the subdomains include Parallelism; Plane Euclidean Geometry, Three-Dimensional Geometry; and Transformational Geometry. These subdomains and their specific content also reflect the organization of the student academic content standards.

In addition to the content specified within the domains, there are also statements about the subject matter skills and abilities applicable to the Mathematics content domains. For example, single subject candidates are expected to be able, among other skills, to use inductive and deductive reasoning to develop, analyze, draw conclusions, and validate conjectures and arguments; construct proofs using contradictions; know the interconnections among mathematics ideas; use techniques and concepts from different domains and subdomains to model the same problem; explain mathematical interconnections with other disciplines, and communicate their mathematical thinking clearly and coherently to others. Similar to the Multiple Subjects SMRs, it

is in describing the subject matter skills and abilities expected of candidates that the single subject SMRs address the area of Mathematical Reasoning.

B. How the Mathematics Subject Matter Requirements Were Established
The SMRS for both multiple subject and single subject teachers were established through a comprehensive development and validation process using subject matter advisory panels of California content experts. The Mathematics SMRs for both the multiple subject credential and the single subject credentials (Foundational Level and full Mathematics) were adopted by the Commission to serve as the basis for the development of CSET mathematics examination items and for the parallel development of mathematics subject matter preparation program standards.

C. How Candidate Subject Matter Competence in Mathematics is Assessed
The subject matter competence of multiple subject teacher candidates is assessed through the California Subject Examinations for Teachers: Multiple Subjects (CSET: MS) Examination, which addresses all subjects covered by the multiple subject credential, including mathematics. For single subject teacher candidates, there are two ways to assess subject matter competence in mathematics: candidates must either pass the CSET: Mathematics Examination (Foundational Level or full Mathematics Examination), or they must complete a Commission-approved subject matter preparation program in Mathematics.

These routes to establishing subject matter competence are in alignment with the No Child Left Behind (NCLB) state compliance plan adopted by the State Board of Education. Teachers demonstrating their subject matter competence by any of these routes are deemed to be NCLB compliant. Prior to the adoption of the state NCLB compliance plan, multiple subject candidates also had the option of completing a Commission-approved elementary subject matter preparation program. This route was discontinued since the state's NCLB compliance plan allowed only the examination route to establishing subject matter competence for multiple subject candidates.

Assessing Candidate Competence in Mathematics via the Examination Route
This section of the agenda item examines how the subject matter knowledge of candidates is assessed through the California Subject Matter Examinations for Teachers (CSET) Multiple Subjects Examination and the CSET: Mathematics Examination.

1. The CSET: MS Examination
The test structure for the CSET: MS Examination is provided in the table below. Mathematics is assessed along with science in Subtest II. The test structure of the CSET:MS was developed to provide an overall balance among the multiple subjects that an elementary teacher is responsible for teaching. Within that structure, individual test items are written to tap into multiple constructs in order to provide the broadest possible coverage of the SMRs and to allow for a reasonable examination testing time for candidates.

CSET: Multiple Subjects

Subtest	Domains	No. of Multiple Choice Questions	No. of Constructed Response Questions (short, focused responses)
I	Reading, Language, and Literature	26	2
	History and Social Science	26	2
	Subtest Total	52	4
II	Science	26	2
	Mathematics	26	2
	Subtest Total	52	4
III	Physical Education	13	1
	Human Development	13	1
	Visual and Performing Arts	13	1
	Subtest Total	39	3

The CSET: MS Examination is offered six times per year within a five-hour test session: candidates may take one or more subtests, including the entire examination, during that time frame. Within Subtest II, there are 52 multiple-choice questions and 4 short-answer constructed-response items that together measure the content areas of Science and Mathematics. Basic four-function calculators are provided for examinees taking Multiple Subjects Subtest II. Directions for calculator use are not provided at the test administration. Examinees may not bring their own calculator for CSET: Multiple Subjects Subtest II.

Candidates must obtain a scaled score of 220 on a scale of 100-300 to pass this subtest. Scores on the CSET: MS Examination are valid for a period of five years for credentialing purposes. A study conducted during 2008 by the Commission's testing contractor for the 2006-2008 testing years showed that candidates were scoring approximately equally in mathematics and science within the single subtest, and that candidates were not using higher science subtest performance to compensate for lower math subtest performance, or vice versa.

Next Review of the CSET: MS Examination
In order for the Commission to maintain viable, legally defensible examinations, the content of these examinations must be periodically reviewed as part of a validity study that ensures that the examination reflects the most current K-12 standards, frameworks, and other relevant documents. The next review of the CSET: MS Examination is scheduled for 2011-12 in alignment with the release of revised state frameworks.

2. The CSET: Mathematics Examination
The CSET: Mathematics Examination consists of three separate subtests, each composed of both multiple choice and constructed response questions. Each subtest is scored separately. The passing standard adopted by the Commission is a scaled score of 220 on a scale of 100-300 on each subtest. The structure of the examination is shown in the table below.

CSET: Mathematics*

Subtest	Domains	No. of Multiple Choice Questions	No. of Constructed Response Items (short focused response)
I	Algebra	24	3
	Number Theory	6	1
	Subtest Total	30	4
II	Geometry	22	3
	Probability and Statistics	8	1
	Subtest Total	30	4
III	Calculus	26	3
	History of Mathematics	4	1
	Subtest Total	30	4

*Candidates verifying subject matter competence by examination for a credential in Foundational-Level Mathematics are required to take and pass Subtests I and II only.

The CSET: Mathematics Examination is offered six times per year within a five-hour test session: candidates may take one or more subtests, including the entire examination, during that time frame. A calculator is allowed only for Mathematics Subtest II: Geometry; Probability and Statistics. Candidates must bring their own graphing calculator to the test administration, and it must be one of the approved models listed in the current version of the CSET registration bulletin. Test administration staff clear the memory of candidates' calculators before and after the test.

Next Review of the CSET: Mathematics Examination
In order for the Commission to maintain viable, legally defensible examinations, the content of these examinations must be periodically reviewed as part of a validity study that ensures that the examination reflects the most current K-12 standards, frameworks, and other relevant documents.
The next review of the CSET: Mathematics Examination is scheduled for 2012, to align with the revised state framework for mathematics which is due in 2011.

Assessing Candidate Competence in Mathematics via the Mathematics Subject Matter Program Completion Route
The content of both the subject matter examinations and the subject matter program standards are based on the same SMRs. Program standards, however, are broader than the SMRs, and include a range of requirements that program sponsors must meet such as support for the program, the nature of the coursework and field experiences to be provided within the program, the qualifications of the faculty in the program, and others.

Within the subject matter program standards, the SMRs, or "Candidate Competencies," serve to describe the mathematics knowledge, skills and abilities the program is expected to assure that the candidates demonstrate prior to program completion. By tying both the examination content and the subject matter preparation standards to the same set of expected mathematics knowledge, skills and abilities, equivalence is assured between the examination route and the subject matter preparation program route for candidates as a means of determining candidates' subject matter competence.

The Mathematics Subject Matter Program Standards were developed by the same content expert advisory panel that worked on the development of the CSET Examinations.

How Subject Matter Preparation Programs Assess Candidate Competence

The single subject matter preparation program standards for mathematics include one standard in particular that addresses how the program will assess the subject matter competence of its candidates. This standard is reprinted for reference below.

> ### Standard 7: Assessment of Subject Matter Competence
>
> The program uses formative and summative multiple measures to assess the subject matter competence of each candidate. The scope and content of each candidate's assessment is consistent with the content of the subject matter requirements of the program and with institutional standards for program completion.

This standard allows programs to determine their own methods of assessing the subject matter competence of teacher candidates as appropriate to the program design. Programs must assure that the assessments are congruent with the subject matter requirements adopted by the Commission, and programs must assure that candidates are provided with clear descriptions of the assessment scope, process and criteria when they start the program. These assessments are not required to be standardized across or within programs, and different assessments may be used with different candidates within a given program. Although the candidate assessment practices addressed by the standards are within the program's discretion, they are, however, subject to review and approval by the Commission via the standards submission review process and the subsequent accreditation processes.

Next Review of the Mathematics Subject Matter Program Standards

The Mathematics Subject Matter Program Standards are scheduled to be reviewed at the same time as the CSET: Multiple Subjects and Single Subjects Examinations (2012).

III. Next Steps

The information in this agenda item will be presented to the Mathematics Advisory Panel as they review current credential requirements to teach mathematics. The next agenda item in this series of information items will look at how teachers are prepared to teach mathematics within the teacher preparation program, and how their pedagogical knowledge about teaching mathematics is assessed.

APPENDIX A
Mathematics Knowledge, Skills and Abilities for Multiple Subject Teachers

Introduction

A defined set of mathematics knowledge, skills and abilities constitutes the expected subject matter Competence in mathematics for teacher candidates. This set of defined mathematics knowledge, skills, and abilities, known as "SMRs," specify the range of mathematics content eligible for inclusion in CSET: MS and CSET: Single Subject Mathematics examination items, and also form the basis for the descriptive candidate Competence statements contained within the subject matter preparation program standards.

The SMRs for multiple subject and single subject mathematics were identified during the SB 2042 reform process (1999-2004), when the Commission redeveloped both subject matter preparation program standards and teacher preparation program standards. Over a period of five years and within three major development phases, subject matter preparation program standards in all credential areas were redeveloped and readopted by the Commission. This process was carried out in tandem with developing and implementing the California Subject Examinations for Teachers (CSET), which took the place of the prior set of subject matter examinations.

The most recent set of mathematics SMRs for multiple subject teachers was developed in 2001 in alignment with California's adopted K-12 student academic content standards in mathematics. The most recent set of mathematics SMRs for single subject teachers (Foundational Level Mathematics and Mathematics) was developed in 2003, also in alignment with California's K-12 student academic content standards in mathematics.

Mathematics Knowledge, Skills and Abilities for Multiple Subject Teachers

Reprinted below are the Mathematics Subject Matter Requirements for the CSET: MS Mathematics/Science Subtest. All mathematics-related examination items for the CSET: MS are drawn from this range of content.

> **1. Content Domains for Subject Matter Understanding and Skill in Mathematics**
>
> **Domain 1:   Number Sense**
>
> **1.1   Numbers, Relationships Among Numbers, and Number Systems.** Candidates for Multiple Subject Teaching Credentials understand base ten place value, number theory concepts (e.g., greatest common factor), and the structure of the whole, integer, rational, and real number systems. They order integers, mixed numbers, rational numbers (including fractions, decimals, and percents) and real numbers. They represent numbers in exponential and scientific notation. They describe the relationships between the algorithms for addition, subtraction, multiplication, and division.   They understand properties of number systems and their relationship to the algorithms, [e.g., 1 is the multiplicative identity; $27 + 34 = 2 \times 10 + 7 + 3 \times 10 + 4 = (2 + 3) \times 10 + (7 + 4)$]. Candidates perform operations with positive, negative, and fractional exponents, as they apply to whole numbers and fractions.
>
> **1.2   Computational Tools, Procedures, and Strategies.** Candidates demonstrate fluency in standard algorithms for computation and evaluate the correctness of nonstandard algorithms.

They demonstrate an understanding of the order of operations. They round numbers, estimate the results of calculations, and place numbers accurately on a number line. They demonstrate the ability to use technology, such as calculators or software, for complex calculations.

Domain 2: Algebra and Functions

2.1 **Patterns and Functional Relationships.** Candidates represent patterns, including relations and functions, through tables, graphs, verbal rules, or symbolic rules. They use proportional reasoning such as ratios, equivalent fractions, and similar triangles, to solve numerical, algebraic, and geometric problems.

2.2 **Linear and Quadratic Equations and Inequalities.** Candidates are able to find equivalent expressions for equalities and inequalities, explain the meaning of symbolic expressions (e.g., relating an expression to a situation and vice versa), find the solutions, and represent them on graphs. They recognize and create equivalent algebraic expressions (e.g., $2(a+3) = 2a + 6$), and represent geometric problems algebraically (e.g., the area of a triangle). Candidates have a basic understanding of linear equations and their properties (e.g., slope, perpendicularity); the multiplication, division, and factoring of polynomials; and graphing and solving quadratic equations through factoring and completing the square. They interpret graphs of linear and quadratic equations and inequalities, including solutions to systems of equations.

Domain 3: Measurement and Geometry

3.1 **Two- and Three-dimensional Geometric Objects.** Candidates for Multiple Subject Teaching Credentials understand characteristics of common two- and three-dimensional figures, such as triangles (e.g., isosceles and right triangles), quadrilaterals, and spheres. They are able to draw conclusions based on the congruence, similarity, or lack thereof, of two figures. They identify different forms of symmetry, translations, rotations, and reflections. They understand the Pythagorean theorem and its converse. They are able to work with properties of parallel lines.

3.2 **Representational Systems, Including Concrete Models, Drawings, and Coordinate Geometry.** Candidates use concrete representations, such as manipulatives, drawings, and coordinate geometry to represent geometric objects. They construct basic geometric figures using a compass and straightedge, and represent three-dimensional objects through two-dimensional drawings. They combine and dissect two- and three-dimensional figures into familiar shapes, such as dissecting a parallelogram and rearranging the pieces to form a rectangle of equal area.

3.3 **Techniques, Tools, and Formulas for Determining Measurements.** Candidates estimate and measure time, length, angles, perimeter, area, surface area, volume, weight/mass, and temperature through appropriate units and scales. They identify relationships between different measures within the metric or customary systems of measurements and estimate an equivalent measurement across the two systems. They calculate perimeters and areas of two-dimensional objects and surface areas and volumes of three-dimensional objects. They relate proportional reasoning to the construction of scale drawings or models. They use measures such as miles per hour to analyze and solve problems.

Domain 4: Statistics, Data Analysis, and Probability

4.1 **Collection, Organization, and Representation of Data.** Candidates represent a collection of data through graphs, tables, or charts. They understand the mean, median, mode,

and range of a collection of data. They have a basic understanding of the design of surveys, such as the role of a random sample.

4.2   **Inferences, Predictions, and Arguments Based on Data.** Candidates interpret a graph, table, or chart representing a data set. They draw conclusions about a population from a random sample, and identify potential sources and effects of bias.

4.3   **Basic Notions of Chance and Probability.** Candidates can define the concept of probability in terms of a sample space of equally likely outcomes. They use their understanding of complementary, mutually exclusive, dependent, and independent events to calculate probabilities of simple events. They can express probabilities in a variety of ways, including ratios, proportions, decimals, and percents.

2. Subject Matter Skills and Abilities Applicable to the Content Domains in Mathematics

Candidates for Multiple Subject Teaching Credentials identify and prioritize relevant and missing information in mathematical problems. They analyze complex problems to identify similar simple problems that might suggest solution strategies. They represent a problem in alternate ways, such as words, symbols, concrete models, and diagrams, to gain greater insight. They consider examples and patterns as means to formulating a conjecture.

Candidates apply logical reasoning and techniques from arithmetic, algebra, geometry, and probability/statistics to solve mathematical problems. They analyze problems to identify alternative solution strategies. They evaluate the truth of mathematical statements (i.e., whether a given statement is always, sometimes, or never true). They apply different solution strategies (e.g., estimation) to check the reasonableness of a solution. They demonstrate that a solution is correct.

Candidates explain their mathematical reasoning through a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and concrete models. They use appropriate mathematical notation with clear and accurate language. They explain how to derive a result based on previously developed ideas, and explain how a result is related to other ideas.

APPENDIX B
Mathematics Knowledge, Skills and Abilities for Single Subject Teachers

The following set of mathematics subject matter requirements are contained within the Commission's adopted Mathematics Single Subject Matter Program Standards (2004).

Part I: Content Domains for Subject Matter Understanding and Skill in Mathematics

Domain 1.   Algebra
Candidates demonstrate an understanding of the foundations of the algebra contained in the Mathematics Content Standards for California Public Schools (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of algebra and its underlying structures, candidates have a deep conceptual knowledge. They are skilled at symbolic reasoning and use algebraic skills and concepts to model a variety of problem-solving situations. They understand the power of mathematical abstraction and symbolism.

1.1   Algebraic Structures
a. Know why the real and complex numbers are each a field, and that particular rings are not fields (e.g., integers, polynomial rings, matrix rings)
b. Apply basic properties of real and complex numbers in constructing mathematical arguments (e.g., if $a < b$ and $c < 0$, then $ac > bc$)
c. Know that the rational numbers and real numbers can be ordered and that the complex numbers cannot be ordered, but that any polynomial equation with real coefficients can be solved in the complex field

(Mathematics Content Standards for California Public Schools, Grade 6, Number Sense: 1.0, 2.0; Grade 7, Algebra and Functions: 1.0; Algebra I: 1.0, 3.0-7.0, 9.0-15.0, 24.0, 25.0; Geometry: 1.0, 17.0; Algebra II: 1.0-8.0, 11.0, 24.0, 25.0; Trigonometry: 17.0; Mathematical Analysis: 2.0; Linear Algebra: 9.0, 11.0)

1.2   Polynomial Equations and Inequalities
a. Know why graphs of linear inequalities are half planes and be able to apply this fact (e.g., linear programming)
b. Prove and use the following:
  - The Rational Root Theorem for polynomials with integer coefficients
  - The Factor Theorem
  - The Conjugate Roots Theorem for polynomial equations with real coefficients
  - The Quadratic Formula for real and complex quadratic polynomials
  - The Binomial Theorem

c. Analyze and solve polynomial equations with real coefficients using the Fundamental Theorem of Algebra
(Mathematics Content Standards for California Public Schools, Grade 7, Algebra and Functions: 2.0-4.0; Algebra I: 1.0, 2.0, 4.0-10.0, 12.0-15.0, 17.0-23.0; Algebra II: 2.0-11.0, 16.0, 17.0; Trigonometry: 17.0, 18.0; Mathematical Analysis: 4.0, 6.0)

1.3 Functions
a. Analyze and prove general properties of functions (i.e., domain and range, one-to-one, onto, inverses, composition, and differences between relations and functions)
b. Analyze properties of polynomial, rational, radical, and absolute value functions in a variety of ways (e.g., graphing, solving problems)

c. Analyze properties of exponential and logarithmic functions in a variety of ways (e.g., graphing, solving problems)

(<u>Mathematics Content Standards for California Public Schools,</u> Grade 6, Algebra and Functions: 1.0; Grade 7, Number Sense: 1.0, 2.0; Algebra and Functions: 3.0; Algebra I: 3.0-6.0, 10.0, 13.0, 15.0-18.0, 21.0-23.0; Algebra II: 1.0-4.0, 6.0-17.0, 24.0, 25.0; Trigonometry: 2.0, 4.0-8.0, 19.0; Mathematical Analysis: 6.0, 7.0; Calculus: 9.0)

1.4   Linear Algebra
a. Understand and apply the geometric interpretation and basic operations of vectors in two and three dimensions, including their scalar multiples and scalar (dot) and cross products
b. Prove the basic properties of vectors (e.g., perpendicular vectors have zero dot product)
c. Understand and apply the basic properties and operations of matrices and determinants (e.g., to determine the solvability of linear systems of equations)

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 9.0; Algebra II: 2.0; Mathematical Analysis: 1.0; Linear Algebra: 1.0-12.0)

Domain 2.   Geometry
Candidates demonstrate an understanding of the foundations of the geometry contained in the <u>Mathematics Content Standards for California Public Schools</u> (1997) as outlined in the <u>Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve</u> (1999) from an advanced standpoint. To ensure a rigorous view of geometry and its underlying structures, candidates have a deep conceptual knowledge. They demonstrate an understanding of axiomatic systems and different forms of logical arguments. Candidates understand, apply, and prove theorems relating to a variety of topics in two- and three-dimensional geometry, including coordinate, synthetic, non-Euclidean, and transformational geometry.

2.1   Parallelism
a. Know the Parallel Postulate and its implications, and justify its equivalents (e.g., the Alternate Interior Angle Theorem, the angle sum of every triangle is 180 degrees)
b. Know that variants of the Parallel Postulate produce non-Euclidean geometries (e.g., spherical, hyperbolic)

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 8.0, 24.0; Geometry: 1.0-3.0, 7.0, 13.0)

2.2   Plane Euclidean Geometry
a. Prove theorems and solve problems involving similarity and congruence
b. Understand, apply, and justify properties of triangles (e.g., the Exterior Angle Theorem, concurrence theorems, trigonometric ratios, Triangle Inequality, Law of Sines, Law of Cosines, the Pythagorean Theorem and its converse)
c. Understand, apply, and justify properties of polygons and circles from an advanced standpoint (e.g., derive the area formulas for regular polygons and circles from the area of a triangle)
d. Justify and perform the classical constructions (e.g., angle bisector, perpendicular bisector, replicating shapes, regular n-gons for n equal to 3, 4, 5, 6, and 8)
e. Use techniques in coordinate geometry to prove geometric theorems
(<u>Mathematics Content Standards for California Public Schools,</u> Grade 6, Algebra and Functions: 2.0, 3.0; Measurement and Geometry: 2.0; Grade 7, Measurement and Geometry: 1.0-3.0;

Algebra I: 8.0, 24.0; Geometry: 1.0-6.0, 8.0-16.0, 18.0-21.0; Algebra II: 16.0, 17.0; Trigonometry: 12.0-14.0, 18.0, 19.0; Mathematical Analysis: 5.0)

2.3 Three-Dimensional Geometry
a. Demonstrate an understanding of parallelism and perpendicularity of lines and planes in three dimensions
b. Understand, apply, and justify properties of three-dimensional objects from an advanced standpoint (e.g., derive the volume and surface area formulas for prisms, pyramids, cones, cylinders, and spheres)

(Mathematics Content Standards for California Public Schools, Grade 6, Measurement and Geometry: 1.0; Grade 7, Measurement and Geometry: 2.0; Algebra I: 24.0; Geometry: 2.0, 3.0, 12.0, 17.0; Mathematical Analysis: 5.0)

2.4 Transformational Geometry
a. Demonstrate an understanding of the basic properties of isometries in two- and three-dimensional space (e.g., rotation, translation, reflection)
b. Understand and prove the basic properties of dilations (e.g., similarity transformations or change of scale)

(Mathematics Content Standards for California Public Schools, Geometry: 11.0, 22.0)

Domain 3. Number Theory
Candidates demonstrate an understanding of the number theory and a command of the number sense contained in the Mathematics Content Standards for California Public Schools (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of number theory and its underlying structures, candidates have a deep conceptual knowledge. They prove and use properties of natural numbers. They formulate conjectures about the natural numbers using inductive reasoning, and verify conjectures with proofs.

3.1 Natural Numbers
a. Prove and use basic properties of natural numbers (e.g., properties of divisibility)
b. Use the Principle of Mathematical Induction to prove results in number theory
c. Know and apply the Euclidean Algorithm
d. Apply the Fundamental Theorem of Arithmetic (e.g., find the greatest common factor and the least common multiple, show that every fraction is equivalent to a unique fraction where the numerator and denominator are relatively prime, prove that the square root of any number, not a perfect square number, is irrational)

(Mathematics Content Standards for California Public Schools, Grade 6, Number Sense: 2.0; Grade 7, Number Sense: 1.0; Algebra I: 1.0, 2.0, 12.0, 24.0, 25.0; Geometry: 1.0; Algebra II: 21.0, 23.0, 25.0; Mathematical Analysis: 3.0)

Domain 4. Probability and Statistics
Candidates demonstrate an understanding of the statistics and probability distributions for advanced placement statistics contained in the Mathematics Content Standards for California

Public Schools (1997) as outlined in the <u>Mathematics Framework for California Public Schools:</u> <u>Kindergarten Through Grade Twelve</u> (1999) from an advanced standpoint. To ensure a rigorous view of probability and statistics and their underlying structures, candidates have a deep conceptual knowledge. They solve problems and make inferences using statistics and probability distributions.

4.1   Probability
a. Prove and apply basic principles of permutations and combinations
b. Illustrate finite probability using a variety of examples and models (e.g., the fundamental counting principles)
c. Use and explain the concept of conditional probability
d. Interpret the probability of an outcome
e. Use normal, binomial, and exponential distributions to solve and interpret probability problems

(<u>Mathematics Content Standards for California Public Schools,</u> Grade 6, Statistics, Data Analysis, and Probability: 3.0; Algebra II: 18.0-20.0; Probability and Statistics: 1.0-4.0; Advanced Probability and Statistics: 1.0-4.0, 7.0, 9.0, 17.0, 18.0)

4.2   Statistics
a.   Compute and interpret the mean, median, and mode of both discrete and continuous distributions
b.   Compute and interpret quartiles, range, variance, and standard deviation of both discrete and continuous distributions
c.   Select and evaluate sampling methods appropriate to a task (e.g., random, systematic, cluster, convenience sampling) and display the results
d.   Know the method of least squares and apply it to linear regression and correlation
e.   Know and apply the chi-square test

(<u>Mathematics Content Standards for California Public Schools,</u> Grade 6, Statistics, Data Analysis, and Probability: 1.0, 2.0; Grade 7, Statistics, Data Analysis, and Probability: 1.0; Probability and Statistics: 5.0-7.0; Advanced Probability and Statistics: 4.0-6.0, 8.0, 10.0-13.0, 15.0-17.0, 19.0)

Domain 5.   Calculus*
Candidates demonstrate an understanding of the trigonometry and calculus contained in the <u>Mathematics Content Standards for California Public Schools</u> (1997) as outlined in the <u>Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve</u> (1999) from an advanced standpoint. To ensure a rigorous view of trigonometry and calculus and their underlying structures, candidates have a deep conceptual knowledge. They apply the concepts of trigonometry and calculus to solving problems in real-world situations.

5.1   Trigonometry
a.   Prove that the Pythagorean Theorem is equivalent to the trigonometric identity $\sin^2 x + \cos^2 x = 1$ and that this identity leads to $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$
b.   Prove the sine, cosine, and tangent sum formulas for all real values, and derive special applications of the sum formulas (e.g., double angle, half angle)
*** Domain 5, Calculus, does not apply to requirements for the Foundational-level Credential.**
c.   Analyze properties of trigonometric functions in a variety of ways (e.g., graphing and solving problems)
d.   Know and apply the definitions and properties of inverse trigonometric functions (i.e., arcsin, arccos, and arctan)

e. Understand and apply polar representations of complex numbers (e.g., DeMoivre's Theorem)

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 24.0; Geometry: 3.0, 14.0, 18.0, 19.0; Algebra II: 24.0, 25.0; Trigonometry: 1.0-6.0, 8.0-11.0, 19.0; Mathematical Analysis: 1.0, 2.0; Calculus: 18.0, 20.0)

5.2   Limits and Continuity
a. Derive basic properties of limits and continuity, including the Sum, Difference, Product, Constant Multiple, and Quotient Rules, using the formal definition of a limit
b. Show that a polynomial function is continuous at a point
c. Know and apply the Intermediate Value Theorem, using the geometric implications of continuity

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 24.0; Geometry: 3.0; Algebra II: 1.0, 15.0; Mathematical Analysis: 8.0; Calculus: 1.0-4.0)

5.3   Derivatives and Applications
a. Derive the rules of differentiation for polynomial, trigonometric, and logarithmic functions using the formal definition of derivative
b. Interpret the concept of derivative geometrically, numerically, and analytically (i.e., slope of the tangent, limit of difference quotients, extrema, Newton's method, and instantaneous rate of change)
c. Interpret both continuous and differentiable functions geometrically and analytically and apply Rolle's Theorem, the Mean Value Theorem, and L'Hopital's rule
d. Use the derivative to solve rectilinear motion, related rate, and optimization problems
e. Use the derivative to analyze functions and planar curves (e.g., maxima, minima, inflection points, concavity)
f. Solve separable first-order differential equations and apply them to growth and decay problems

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 5.0-8.0, 10.0, 11.0, 13.0, 21.0, 23.0; Geometry: 3.0; Algebra II: 1.0, 9.0, 10.0, 12.0, 15.0; Trigonometry: 7.0, 15.0-19.0; Mathematical Analysis: 5.0, 7.0; Calculus: 1.0, 4.0-12.0, 27.0)

5.4   Integrals and Applications
a. Derive definite integrals of standard algebraic functions using the formal definition of integral
b. Interpret the concept of a definite integral geometrically, numerically, and analytically (e.g., limit of Riemann sums)
c. Prove the Fundamental Theorem of Calculus, and use it to interpret definite integrals as antiderivatives
d. Apply the concept of integrals to compute the length of curves and the areas and volumes of geometric figures

(<u>Mathematics Content Standards for California Public Schools,</u> Algebra I: 24.0; Geometry: 9.0; Calculus: 13.0-23.0)

5.5   Sequences and Series
a. Derive and apply the formulas for the sums of finite arithmetic series and finite and infinite geometric series (e.g., express repeating decimals as a rational number)
b. Determine convergence of a given sequence or series using standard techniques (e.g., Ratio, Comparison, Integral Tests)
c. Calculate Taylor series and Taylor polynomials of basic functions

(Mathematics Content Standards for California Public Schools, Algebra I: 24.0, 25.0; Algebra II: 21.0-23.0; Mathematical Analysis: 8.0; Calculus: 23.0-26.0)

Domain 6.   History of Mathematics*
Candidates understand the chronological and topical development of mathematics and the contributions of historical figures of various times and cultures. Candidates know important mathematical discoveries and their impact on human society and thought. These discoveries form a historical context for the content contained in the Mathematics Content Standards for California Public Schools (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999; e.g., numeration systems, algebra, geometry, calculus).

6.1   Chronological and Topical Development of Mathematics
a.   Demonstrate understanding of the development of mathematics, its cultural connections, and its contributions to society
b.   Demonstrate understanding of the historical development of mathematics, including the contributions of diverse populations as determined by race, ethnicity, culture, geography, and gender

*Domain 6, History of Mathematics, does not apply to requirements for the Foundational-level Credential.

Part II: Subject Matter Skills and Abilities Applicable to the Content Domains in Mathematics

(All elements of Part II apply to both the Single Subject Credential in Mathematics and the Single Subject Credential in Foundational Mathematics.)

Candidates for Single Subject Teaching Credentials in mathematics use inductive and deductive reasoning to develop, analyze, draw conclusions, and validate conjectures and arguments. As they reason, they use counterexamples, construct proofs using contradictions, and create multiple representations of the same concept. They know the interconnections among mathematical ideas, and use techniques and concepts from different domains and sub-domains to model the same problem. They explain mathematical interconnections with other disciplines. They are able to communicate their mathematical thinking clearly and coherently to others, orally, graphically, and in writing, through the use of precise language and symbols.

Candidates solve routine and complex problems by drawing from a variety of strategies while demonstrating an attitude of persistence and reflection in their approaches. They analyze problems through pattern recognition and the use of analogies. They formulate and prove conjectures, and test conclusions for reasonableness and accuracy. They use counterexamples to disprove conjectures.

Candidates select and use different representational systems (e.g., coordinates, graphs). They understand the usefulness of transformations and symmetry to help analyze and simplify problems. They make mathematical models to analyze mathematical structures in real contexts. They use spatial reasoning to model and solve problems that cross disciplines.

(Mathematics Content Standards for California Public Schools, Grade 6, Mathematical Reasoning: 1.0-3.0; Grade 7, Mathematical Reasoning: 1.0-3.0)

APPENDIX C
Mathematics Subject Matter Program Standards

(note: The material reprinted below represents excerpts from the full set of program standards)

Standard 11: Required Subjects of Study

In the program, each prospective teacher studies and learns advanced mathematics that incorporates the <u>Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve</u> (1997) and the <u>Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve</u> (1999). The curriculum of the program addresses the *Subject Matter Requirements* and standards of program quality as set forth in this document.

Required Elements

11.1*   Required coursework includes the following major subject areas of study: algebra, geometry, number theory, calculus, history of mathematics, and statistics and probability. This coursework also incorporates the content of the student academic content standards from an advanced viewpoint (see *Attachment to Standard 11: Required Subjects of Study* page 18). Furthermore, infused in required coursework are connections to the middle school and high school curriculum.

11.2   Required coursework exposes underlying mathematical reasoning, explores connections among the branches of mathematics, and provides opportunities for problem solving and mathematical communication.

11.3   Required courses are applicable to the requirements for a major in mathematics. Remedial classes and other studies normally completed in K-12 schools are not counted in satisfaction of the required subjects of study.

11.4   The institution that sponsors the program determines, establishes and implements a standard of minimum scholarship for coursework in the program.

11.5   Required coursework includes work in computer science and/or related mathematics such as: 1) discrete structures (sets, logic, relations and functions) and their application in the design of data structures and programming; 2) design and analysis of algorithms including the use of recursion and combinations; and, 3) use of the computer applications and other technologies to solve problems.

*Calculus and history of mathematics are not required subjects of study for the foundational-level credential.

Standard 12: Problem Solving

In the program, prospective teachers of mathematics develop effective strategies for solving problems both within the discipline of mathematics and in applied settings that include non-routine situations. Problem-solving challenges occur throughout the program of subject matter preparation in mathematics. Through coursework in the program, prospective teachers develop a sense of inquiry and perseverance in solving problems.

Required Elements

In the program, each prospective teacher learns and demonstrates the ability to:

12.1   Place mathematical problems in context and explore their relationship with other problems.

12.2   Solve mathematical problems in more than one way when possible.

12.3   Generalize mathematical problems in more than one way when possible.

12.4   Use appropriate technologies to conduct investigations and solve problems.

Standard 13: Mathematics as Communication

In the program, prospective teachers learn to communicate their thinking clearly and coherently to others using appropriate language, symbols and technologies. Prospective teachers develop communication skills in conjunction with mathematical literacy in each major component of a subject matter program.

Required Elements

In the program, each prospective teacher learns and demonstrates the ability to:

13.1   Articulate mathematical ideas verbally and in writing, using appropriate terminology.

13.2   Where appropriate present mathematical explanations suitable to a variety of grade levels.

13.3   Present mathematical information in various forms, including but not limited to models, charts, graphs, tables, figures, and equations.

13.4   Analyze and evaluate the mathematical thinking and strategies of others.

13.5   Use clarifying and extending questions to learn and to communicate mathematical ideas.

13.6   Use appropriate technologies to present mathematical ideas and concepts.

Standard 14: Reasoning

In the program, prospective teachers of mathematics learn to understand that reasoning is fundamental to knowing and doing mathematics. Reasoning and proof accompany all mathematical activities in the program.

Required Elements

In the program, each prospective teacher learns and demonstrates the ability to:

14.1   Formulate and test conjectures using inductive reasoning, construct counter-examples, make valid deductive arguments, and judge the validity of mathematical arguments in each content domain of the subject matter requirements.

14.2   Present informal and formal proofs in oral and written formats in each content domain of the subject matter requirements.

Standard 15: Mathematical Connections

In the program, prospective teachers of mathematics develop a view of mathematics as an integrated whole, seeing connections across different mathematical content areas. Relationships among mathematical subjects and applications are a consistent theme of the subject matter program's curriculum[1].

Required Elements

In the program, each prospective teacher learns and demonstrates the ability to:

15.1   Illustrate, when possible, abstract mathematical concepts using applications.

15.2   Investigate ways mathematical topics are inter-related.

15.3   Apply mathematical thinking and modeling to solve problems that arise in other disciplines.

15.4   Recognize how a given mathematical model can represent a variety of situations.

15.5   Create a variety of models to represent a single situation.

15.6   Understand the interconnectedness of topics in mathematics from an historical perspective.

Standard 16: Delivery of Instruction

In the program, faculty use multiple instructional strategies, activities and materials that are appropriate for effective mathematics instruction.

Required Elements

Coursework in the program:

16.1   Is taught in a way that fosters conceptual understanding as well as procedural knowledge.

16.2   Incorporates a variety of instructional formats including but not limited to direct instruction, collaborative groups, individual exploration, peer instruction, and whole class discussion led by students.

16.3   Provides for learning mathematics in different modalities, e.g., visual, auditory, and kinesthetic.

16.4   Develops and reinforces mathematical skills and concepts through open-ended activities.

16.5   Uses a variety of appropriate technologies.

16.6   Includes approaches that are appropriate for use at a variety of grade levels.

Attachment to Standard 11:

Required Subjects of Study

The main purpose of the Subject Matter Requirements (SMRs) is to provide a guideline for the education of prospective mathematics teachers so that they will be well equipped to teach to the state-adopted Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997), and that they have a mathematical understanding and proficiency beyond those Standards. Taken at face value, the SMRs define a minimum core of skills, abilities, and understandings for all candidates of the Single Subject Teaching Credential in Mathematics. Ideally, teacher candidates develop an advanced viewpoint of the content areas represented in this core. The intent of this attachment is to give a sense of the mathematical context in which such advanced viewpoints can be developed. The attachment provides examples and ideas for this development, and is not intended to be prescriptive. While some of these examples may seem obvious to a professor of mathematics, many mathematics majors do not make the connections. Therefore, these ideas are important for prospective teachers.

It is important to note three principles that guided the development of the SMRs:
      a) mathematical reasoning is central to mathematical understanding;
      b) mathematics requires knowledge that is connected and integrated; and
      c) college faculty are central to shaping the curriculum of subject matter programs.

First, the emphasis on mathematical reasoning amplifies what is already clearly enunciated in a critical passage of the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999; Framework):

> From kindergarten through grade 7, these [content] standards have impressed on the students the importance of logical reasoning in mathematics. Starting with grade 8, students should be ready for the basic message that logical reasoning is the underpinning of all of mathematics. In other words, every assertion can be justified by logical deduction from previously known facts. Students should begin to learn to prove every statement that they make. Every textbook or mathematics lesson should strive to convey this message, and to convey it well. (p. 154)

In order for such a vision of mathematics education to materialize, teachers themselves need to be well versed in writing proofs and explaining them. For this reason, the SMRs emphasize logical explanations, and formal and informal proofs. Explanations and proofs also underscore the fact that logical arguments occur not only in Euclidean geometry but everywhere.

A proof is a logical explanation of why a statement holds. It need not have any particular form, and the emphasis should be on the student understanding why a result holds. Written proofs in textbooks may serve as a model for exposition, but never as a model for the discovery of a proof. Proofs are usually found by painstaking trials and errors, and almost never in the logical sequence of steps laid out in written proofs. It should be emphasized that it is the logical correctness of a proof that is important, not the literary polish of the presentation of the proof. The common complaint that geometry proofs in a real classroom have become a ritual divorced from mathematics would disappear if teachers are made more aware of the need to pay attention to mathematical substance rather than minute details of the write-up of a proof. A correct proof can be legitimately presented in many ways (e.g., two-column format, paragraph format, flow-chart format). No one format is inherently superior to any other.

Second, the integration of subject matter is implied in more than a few of the standards. Although the SMRs are divided into separate content domains (e.g., algebra, geometry) such a division is more for the convenience of communication rather than an advocacy for a rigid separation of mathematical instruction. For example, prospective teachers should be able to analyze and solve polynomial equations using the roots of unity. This statement assumes that the prospective teacher understands De Moivre's Theorem (SMR 5.1e) and basic properties of regular polygons. In this case, algebra, trigonometry, and geometry

are completely intermingled. As another example, prospective teachers need to be able to teach the graphing of polynomials, but simple facts about such graphs (e.g., that the graph of an *nth* degree polynomial has at most n-1 "peaks" and "valleys") are not accessible without the use of calculus.

Third, the SMRs are not prescriptive about curriculum or pedagogy. There is plenty of room for the creative and informed judgments of faculty to direct the education of teachers of mathematics. For example, although it is not included in the SMRs, faculty may choose to present the derivation of the cubic formula for the purpose of deepening teachers' understanding and appreciation of the quadratic formula. Similarly, some faculty may view SMR 1.3c, which deals with properties of the logarithm function, as an implicit invitation to go into the origin of the logarithm. Napier's invention of logarithms in the 1600s was the device which, in the word of the French mathematician-astronomer Laplace, "by shortening the labors, doubled the life of the astronomer." When teachers understand this utility, and the parallels of the discovery of logarithms with the discovery and development of computing technologies, they are much better equipped to motivate students' understanding of such mathematical topics.

The following sections provide some ideas and examples for developing an advanced viewpoint, particularly about the importance of mathematical reasoning and connections, through the main subject areas of the SMRs.

Algebra

Mathematical reasoning

Prospective teachers' understanding of the three fields they use most often – rational, real, and complex numbers – should include what it means for rational and real numbers to be ordered fields, and why complex numbers cannot be ordered. Inequalities make sense in real numbers, because they are ordered. However, prospective teachers should understand that although inequalities do not make sense in complex numbers, equations have a fuller role with them, because every polynomial equation with real or complex coefficients can be completely solved in complex numbers by the Fundamental Theorem of Algebra (SMR 1.1c, 1.2c).

Implicit in SMR 1.2a, which calls for a proof of why the graph of a linear inequality is a half plane, is the need for a proof of the fact that the graph of a linear function is a straight line. The latter proof requires the use of basic properties of similar triangles.
The proof of the result that the roots of real polynomials come in complex conjugate pairs (SMR 1.2b) allows one to see how to make use of the Fundamental Theorem of Algebra in a nontrivial way. In the process, one gains a better understanding of both the Fundamental Theorem of Algebra and the Quadratic Formula.

The rational root theorem for polynomials with integer coefficients (SMR 1.2b) is one that students and textbooks often mistake as a recipe for locating all the roots of such a polynomial. By reviewing the proof carefully, a prospective teacher is likely to understand the full meaning of this theorem.

The Binomial Theorem (SMR 1.2b) occupies a place of honor in algebra and has important connections in other areas of mathematics. Prospective teachers should be able to understand one of its most accessible proofs, and thereby learn a substantive application of mathematical induction.

Connections
Although the SMRs are organized into discrete content domains (e.g., algebra or calculus), prospective teachers should learn that these domains cannot be rigidly separated. For example, the importance of the exponential function (SMR 1.3c) stems primarily from the fact that it is the unique solution of the differential equation $f'(x) = f(x)$ with the initial condition $f(0) = 1$ (SMR 5.3f). It should be emphasized

that it is because of this differential equation that the exponential function e^x (exp x) shows up in the growth and decay problems of algebra textbooks.

The fundamental difference between polynomial functions and both exponential and logarithmic functions should be emphasized (SMR 1.3b, c). T he overriding concern with a polynomial is to locate its roots and the roots of its derivative (to get the x-intercepts as well as the "peaks" and "valleys" of its graph). For exponential and logarithmic functions, however, such a concern does not exist because log x has exactly one root whereas exp x has no root at all. Moreover, both are strictly increasing functions; their graphs have no "peaks" or "valleys." Therefore our interests in the latter functions are different in kind. Our interests in the exponential and logarithmic functions are that log x converts multiplication into addition [i.e., log (ab) = log a + log b] while exp x does the opposite [i.e., exp (a+b) = (exp a)(exp b)], and the fact that they are inverses to each other [i.e., log (exp x) = x for all x and exp (log y) = y for all positive y]. The algebraic properties of log x account for its historical importance as a computational aid (logarithm tables). Analytically, it is the fact that exp x is the solution of f'(x) = f(x), as discussed above, and that log x is the function that has derivative 1/x and satisfies log 1 = 0. The trigonometric functions are important for yet a different reason: periodicity (SMR 5.1c). Many natural phenomena are periodic, and their modeling would require the trigonometric functions. Such a conceptual understanding of these three classes of functions is indispensable to helping teachers make sense of the functions they see almost daily in algebra classes.

Although the topic of rationalizing denominators is not one that is seen as essential, it is one for which a strong connection can be made with ideas from an advanced perspective. One example that shows how rationalizing denominators is related to more advanced ideas is the "rationalizing" of the denominator of $\dfrac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2}$, which is to find a polynomial in $2^{\frac{1}{3}}$ with rational coefficients so that multiplying the denominator $4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2$ by this polynomial equals a rational number. Let $x = 2^{\frac{1}{3}}$, then the denominator becomes $x^2 - 2x + 2$. In the polynomial ring, Q[x] (where Q is the field of rational numbers), the polynomials $x^3 - 2$ and $x^2 - 2x + 2$ are relatively prime and therefore, by the Euclidean algorithm, there are polynomials p(x) and q(x) in Q[x] so that p(x)($x^2 - 2x + 2$) + q(x)($x^3 - 2$) = 1. Letting $x = 2^{\frac{1}{3}}$ gives p($2^{\frac{1}{3}}$)($4^{\frac{1}{3}}$-2($2^{\frac{1}{3}}$) + 2) = 1. It turns out that p(x) = $\dfrac{1}{10}$ $(x^2 + 3x + 4)$, so that multiplying the numerator and denominator of $\dfrac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2}$ by p($2^{\frac{1}{3}}$) = $\dfrac{1}{10}(4^{\frac{1}{3}} + 3(2^{\frac{1}{3}}) + 4)$ leads to $\dfrac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2} = \dfrac{1}{10} (4^{\frac{1}{3}} + 3(2^{\frac{1}{3}}) + 4)$. Engaging in this example will help candidates to make a good connection between topics that they studied in their abstract algebra course and ideas related to the high school curriculum.

Geometry

Mathematical Reasoning

The great challenge in a college geometry course for prospective teachers is teaching fluency with informal and formal proofs of geometric theorems in general and theorems in Euclidean geometry (SMR

2.2) in particular. There is a thorough discussion of this issue in Chapter 3 of the 1999 Framework (pp. 162-7; see also Appendix D on pp. 279-296). The following are key points:

(a) One cannot learn how to prove theorems in geometry without any geometric intuition. One way to acquire such an intuition is to perform constructions with a ruler and compass, and to examine many models of standard solids (e.g., cubes, cones, cylinders).

(b) An introductory college geometry course should start from the beginning. One way to gain the confidence of prospective teachers is not to force them to write any proofs until they have been shown many nontrivial proofs of interesting theorems (see Appendix D of the 1999 Framework). Begin slowly, allowing them to imitate some standard proofs before they venture forth on their own. This is analogous to the method of teaching people how to speak a foreign language whereby you have them listen to the language for many hours before asking them to try to speak it.

(c) In middle and high school geometry as well as college-level geometry courses, one should de-emphasize the proofs of simple theorems that come near the beginning of the axiomatic development. The proofs of such theorems are harder to learn than those of theorems that follow, and this is true not only for beginners but also for professional mathematicians as well. These proofs also tend to be tedious and uninspiring. One way to acquaint prospective teachers with the proofs of more substantive theorems as soon as possible is to adopt the method of "local axiomatics," which is to list the facts one needs for a particular proof, and then proceed to construct the proof on the basis of these facts. This approach mirrors the axiomatic method because, in effect, these facts are the "axioms" in this particular setting (see the examples in Appendix D of the 1999 Framework).

Connections

The historical importance of the parallel postulate, not just in geometry but in all of mathematics up to the nineteenth century, should be thoroughly discussed (SMR 2.1a, b). In middle and high school geometry textbooks, this postulate is stated (if it is stated at all) as "through a point not on a given line, there is one and only one line parallel to the given line." The correct formulation replaces the phrase "there is one and only one" with "there is at most one." In other words, while the existence of the parallel line can be proved, the uniqueness must be assumed. This then gives a natural setting to introduce the concept of "uniqueness," which is a difficult concept for many students. In this context, an informal discussion of the counterparts of the parallel postulate in spherical and hyperbolic geometry (SMR 2.1b) will likely clarify the situation.

The deduction of the parallel postulate from the assumption that "every triangle has an angle sum of $180°$" is somewhat more sophisticated than most of the theorems in plane Euclidean geometry, but when done carefully it can be immensely rewarding (SMR 2.1a).

Although the notion of area will be defined using the Riemann integral in the context of calculus (SMR 5.4d), it is essential for the teaching of middle and high school geometry that a basic definition of area be provided for plane geometric figures. From this definition, a prospective teacher should be able to derive the area formulas for regular polygons, and many other plane geometric figures.

The theorem that every polygon can be triangulated into non-overlapping triangles allows the areas of polygons to be calculated once the areas of the triangles are known (SMR 2.2c). There is, however, no analogous theorem for the volume of a general polyhedron (SMR 2.3b). This is because it can be proved (using advanced techniques) that there is no corresponding elementary algorithm to compute the volume of a general (non-regular) tetrahedron from the volume of a cube. Although the proof of this theorem is

too difficult for an introductory course, prospective teachers need to know this fact to be able to explain to their students why all volume formulas (except that of a rectangular prism) require the use of calculus or equivalent limit arguments. However, from a basic definition of volume, with the use of informal arguments and Cavalieri's Principle, the volumes of prisms, pyramids, cones, cylinders, and spheres can be informally derived. Moreover, teachers should be aware that formally, the coefficient 1/3 in the volume formulas of cones and pyramids comes from integrating x^2 (SMR 5.4d).

A key reason for introducing coordinates and discussing geometric transformations (SMR 2.4a, b) is to be able to clarify the concepts of congruence and similarity, not just for triangles or polygons, but for all plane and space figures. In other words, one defines two such figures to be *congruent* if one is the image of the other under an isometry, and defines them to be *similar* if one is the image of the other under an isometry followed by a dilation. Then it can be shown that when the figures are polygons, these concepts coincide with those of the equality of angles and proportionality of sides.

Number Theory

Mathematical Reasoning

The well known divisibility rules for division by 3, 4, 5, 8, or 9 are usually stated and used in middle and high school textbooks but not often explained. It is imperative that prospective teachers understand the simple proofs of these rules (SMR 3.1a).

From the point of view of middle and high school mathematics, there are at least two aspects of the Fundamental Theorem of Arithmetic that are noteworthy. First, a completely correct proof of the existence of a prime decomposition for whole numbers requires the use of complete induction (and this gives an important example of a different application of mathematical induction). Second, whereas in middle and high school mathematics only the existence part of the theorem is used, one discovers that in fact it is the uniqueness of the prime decomposition that is important and difficult to prove. Experience shows that this particular uniqueness statement - more so than the uniqueness in the parallel postulate or the uniqueness of the remainder in the division algorithm - is elusive to beginners. The uniqueness is an essential aspect of the Fundamental Theorem of Arithmetic; otherwise, the proof of the irrationality of 5 (or any whole number not a perfect square) or why every fraction is equivalent to a unique fraction in lowest terms would be meaningless.

Connections
The Euclidean algorithm (SMR 3.1c) requires a strong understanding of the division algorithm, including a clear conceptualization of a remainder, and thus the uniqueness of the remainder in the division algorithm. This is another area in which the content domains merge. Prospective teachers should understand both the division algorithm and the Euclidean algorithm for polynomials with real coefficients, and the relationship to the results in number theory.

Calculus*

Mathematical Reasoning
One should emphasize that the sine and cosine addition theorems are the defining theorems of trigonometry (SMR 5.1b). Indeed, it can be proved that sine and cosine are the only differentiated functions satisfying the addition theorems and the condition that sin 0 = 0 and cos 0 = 1. Moreover, every trigonometric identity is a consequence of these addition theorems, and the identity that

$\sin^2 x + \cos^2 x = 1$. Thus the latter identity and the addition theorems are the foundation of trigonometry. This fact gives structure to the subject, and should be clearly understood by each prospective teacher.

In the teaching of calculus, it would be inappropriate to insist on epsilon-delta proofs, but it would be equally inappropriate to eliminate such proofs altogether. Therefore, SMR 5.2 requires that at least the correct definition of limit be provided and applied in a restricted way. This can be accomplished by proving the continuity of quadratic polynomials using epsilon-delta. One benefit of this insistence on a minimal amount of rigor is to expose prospective teachers to the fallacy of the common perception that the continuity of f(x) means "a small change in x produces a small change in f(x)." For instance, if this were the case, should not a change in x to the order of 1/10000 produce a "small" change in f(x)? The answer is, of course, no, because if f(x) = $10^9 x$, then a change in x of 1/10000 produces a change of 100000 in f(x). Thus, one can see why precision in mathematics (such as that found in the tortuous definition of continuity) is necessary. Not insisting on precise proofs on the most common differentiation formulas is likely to invite some abuse. For example, the usual proof "from the product rule of differentiation, one can prove the quotient rule" is a common pitfall that should be avoided, especially in the context of middle and high school mathematics. The putative proof goes as follows: because f(x) (1/f(x)) = 1, differentiating both sides and applying the product rule on the right side of the formula gives f'(x) (1/f(x)) + f(x) (1/f(x))' = 0, from which it follows that (1/f(x))' = - f'(x)/[f(x)^2]. Once this is known, another application of the product rule to g(x)(1/f(x)) gives the usual quotient rule for g(x)/f(x). This is the "proof" of the quotient rule. The fallacy of the preceding argument lies in the fact that until one knows 1/f(x) is differentiated one cannot apply the product rule to f(x)(1/f(x)). Of course, when one tries to prove the differentiation of 1/f(x), the result is the usual messy proof of the quotient rule. What can be claimed is that the above method gives a mnemonic device to remember the quotient rule. Such a statement, when so carefully phrased, has pedagogical value in a calculus classroom, but by no means should one convey the misconception that the product rule proves the quotient rule. Similar comments apply to the differentiation of the square root of a function or, in fact, of any rational power of a function.

The calculus SMRs require the proofs of few theorems, one of which is the proof of the Fundamental Theorem of Calculus (SMR 5.4c). Intended by this SMR is a proof that assumes the basic properties of continuous functions and the integral (e.g., that a continuous function attains a maximum and a minimum on a closed interval, that the integral is linear in the integrand, and that the integral of positive functions is positive). The reason prospective teachers should know this proof is not only that the Fundamental Theorem is truly fundamental (and why this is so should, of course, be carefully explained), but also that this proof is very instructive.

Connections

Both finite and infinite geometric series are important because they appear frequently (SMR 5.5a). In particular, one aspect of infinite geometric series deserves comment, namely the fact that the formal way of summing a geometric series gives rise to the expression of a repeating decimal as a fraction. This mechanism should be conducted carefully as it is often presented incorrectly in middle and high school textbooks. One reason for mentioning the convergence of infinite geometric series (SMR 5.5b) is to make sense of infinite decimals: an infinite decimal is merely a shorthand notation for a particular kind of infinite series. For Taylor series (SMR 5.5c), candidates should know at least the formalism of associating a power series to any one of the elementary functions. Candidates should be able to recognize the sine, cosine, and exponential series.

History of Mathematics

Many important developments in mathematics are too advanced to be discussed in an introductory course on the history of mathematics, yet four major developments that directly impact middle and high school mathematics deserve special attention (SMR 6.1b). The first development is the history of numeral

systems through the early civilizations of Babylon, Rome, and china, and through the so-called Hindu-Arabic decimal system. A second development is the evolution of symbolic algebra, which includes contributions from Diophantus, the Hindus, Viete, and the finishing touches of Descartes. An understanding of this long and uneasy development enhances one's understanding of middle and high school mathematics as a whole. The third development is of calculus, which is rooted in ideas from Eudoxus and Archimedes, the rich but informal development of Newton and Leibniz, and the rigorous formulation that culminated with Cauchy. The fourth and last development is the concept of a proof and, therewith, the concept of an axiomatic system. Proofs formally originated with Euclid's work, and until the twentieth century, were essentially the defining characteristic of European mathematics. For almost two centuries, the questionable foundation of calculus almost forced an abandonment of the classical ideal of proofs in mathematics. It was only toward the end of the nineteenth century when proofs would again occupy center stage and a clear definition of a proof was achieved.