Standards of Program Quality and Effectiveness for Mathematics Subject Matter Programs

Commission on Teacher Credentialing

Standards Adopted
2003

Revised *2010*
to align to Common Core State Standards

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A. Submission Guidelines for Mathematics Subject Matter Preparation Programs

An institution interested in offering a new Commission-approved Mathematics subject matter program must submit the following:

- An Intent to Submit form;
- The Initial Program Review (IPR) Cost Recovery Fee;
- A program proposal that responds to the preconditions and adopted program standards (Appendix B) - limit of 5 pages per standard response;
- A course scope and sequence (typically taken from the institution's catalog);
- A completed Mathematics Alignment Matrix; and
- All of the program's course syllabi.

Further information on the program submission, review process, and cost recovery fees is available on the Subject Matter Program Approval web page: http://www.ctc.ca.gov/educator-prep/elig-inst-new-subject.html.

B. Subject Matter Program Preconditions

Introduction to the Preconditions

A precondition is a requirement for initial subject matter program approval. Unlike standards, preconditions specify requirements for program *compliance* rather than *program quality*. The basis for a precondition is either: 1) statute; 2) regulations; and/or 3) Commission policy. The Commission determines whether a program complies with the adopted preconditions on the basis of a program document provided by the institution for initial program review purposes. In the overall program review sequence, only after a program demonstrates compliance with all preconditions is it then eligible for a more intensive review to determine if the program is in alignment with the Commission's Mathematics subject matter program standards for quality and effectiveness.

Institutions may determine whether the preconditions and standards are addressed through one or more courses for each commonly taught subject or courses offering integrated study of these subjects. Institutions may also define the program in terms of required or elective coursework. However, elective options must be equivalent in meeting the standards. Coursework offered by any appropriate department(s) of a regionally accredited institution may satisfy the preconditions and standards in this handbook. Programs may use general education courses in meeting the standards.

Mathematics Subject Matter Program Preconditions

1. Each Program of Subject Matter Preparation for the Single Subject Teaching Credential in Mathematics shall include (a) a minimum of 30 semester units (or 45 quarter units) of core coursework in Mathematics and related subjects that are commonly taught in departmentalized classes in California public schools, and (b) a minimum of 15 semester units (or 22 quarter units) of coursework that provides extended study of the subject. These two requirements are elaborated in Preconditions 2 and 3.

2. The core (breadth) of the program shall include coursework in (or directly related to) the following subjects that are commonly taught in departmentalized classes of Mathematics and related subjects in the public schools, including:
    - Algebra;
    - Geometry;
    - Number Theory;
    - Statistics and Probability; and
    - Calculus

3. Extended studies in the program (breadth, depth, perspective, concentrations) designed to supplement the core of the program may be offered in any or all of the following patterns:
    - A combination of related content areas within or across domains

- A concentration in one domain
- A concentration in any content area within a domain

Foundational Mathematics Subject Matter Program Preconditions

1. Each Program of Subject Matter Preparation for the Single Subject Teaching Credential in Foundational Mathematics shall include (a) a minimum of 20 semester units (or 30 quarter units) of core coursework in Mathematics and related subjects that are commonly taught in departmentalized classes in California public schools, and (b) a minimum of 12 semester units (or 18 quarter units) of coursework that provides extended study of the subject. These two requirements are elaborated in Preconditions 2 and 3.

2. The core (breadth) of the program shall include coursework in (or directly related to) the following subjects that are commonly taught in departmentalized classes of Mathematics and related subjects in the public schools, including:
    - Algebra;
    - Geometry;
    - Number Theory; and
    - Statistics and Probability

3. Extended studies in the program (breadth, depth, perspective, concentrations) designed to supplement the core of the program may be offered in any or all of the following patterns:
    - A combination of related content areas within or across domains
    - A concentration in one domain
    - A concentration in any content area within a domain

C. Single Subject Matter Preparation Program Standards

Category I: Standards Common to All Single Subject Matter Programs

The following two standards related to program design, resources, and support are applicable to programs in *all* single subject disciplines. They are the same for all subject areas and require subject-specific program information.

Standard 1: Program Design

Subject matter programs are based on an explicit statement expressing the purpose, design, and expected outcomes of the program. The program curriculum builds on the K-12 state-adopted academic content standards, with candidate outcomes and assessments aligned to the subject matter requirements. The program provides prospective teachers with conceptual knowledge of the subject matter, develops academic literacy and discipline-based fluency, addresses issues of equity and diversity, and exposes prospective teachers to a variety of learning experiences appropriate for the discipline.

Standard 2: Program Resources and Support

The program sponsor allocates resources to support effective program coordination, which includes advising students, facilitating collaboration among stakeholders and overseeing program review. Ongoing review processes use assessments of the prospective teachers and a variety of data such as input from stakeholders and other appropriate measurements for review and evaluation of the subject matter program.

Category II: Program Standards for Mathematics

Standard 3: Required Subject of Study
The subject matter preparation program is based on an explicit statement of program philosophy that expresses a purpose and design, and articulates desired outcomes, that are aligned to the Preliminary Multiple and Single Subject Credential Program Standards. The program provides the coursework and field experiences necessary to teach the specified subject to all students in California's diverse public school population. The subject matter preparation for prospective teachers is academically rigorous and intellectually stimulating. The institution assigns high priority to and appropriately supports the program as an essential part of its mission. The program curriculum reflects and builds on the Common Core State Standards for Mathematics (2013) and the Mathematics Framework (2013) adopted by the State Board of Education. The program is designed to establish a strong foundation in subject matter knowledge and understanding that provides a basis for continued development during the teacher's professional career.

Standard 4: Problem Solving
In the program, prospective teachers of mathematics develop effective strategies for solving problems both within the discipline of mathematics and in applied settings that include non-routine situations. Problem solving challenges occur throughout the program of subject matter preparation in mathematics. Through coursework in the program, prospective teachers develop a sense of inquiry and perseverance in solving problems.

Standard 5: Mathematics as Communication
In the program, prospective teachers learn to communicate their thinking clearly and coherently to others using appropriate language, symbols and technologies. Prospective teachers develop communication skills in conjunction with mathematical literacy in each major component of a subject matter program.

Standard 6: Reasoning
In the program, prospective teachers of mathematics learn to understand that reasoning is fundamental to knowing and doing mathematics. Reasoning and proof accompany all mathematical activities in the program.

Standard 7: Mathematical Connections
In the program, prospective teachers of mathematics develop a view of mathematics as an integrated whole, seeing connections across different mathematical content areas. Relationships among mathematical subject and applications are a consistent theme of the subject matter program's curriculum.

Standard 8: Delivery of Instruction
In the program, faculty use multiple instructional strategies, activities and materials that are appropriate for effective mathematics instruction.

D. Mathematics Subject Matter Requirements

Domain 1: Algebra

Candidates demonstrate an understanding of the foundations of algebra as outlined in the California Common Core Content Standards for Mathematics (Grade 7, Grade 8, and High School). Candidates demonstrate a depth and breadth of conceptual knowledge to ensure a rigorous view of algebra and its underlying structures. They are skilled at symbolic reasoning and use algebraic skills and concepts to model a variety of problem-solving situations. They understand the power of mathematical abstraction and symbolism.

1.1 Algebraic Structures

a. Demonstrate knowledge of why the real and complex numbers are each a field, and that particular rings are not fields (e.g., integers, polynomial rings, matrix rings)
b. Apply basic properties of real and complex numbers in constructing mathematical arguments (e.g., if $a < b$ and $c < 0$, then $ac > bc$)
c. Demonstrate knowledge that the rational numbers and real numbers can be ordered and that the complex numbers cannot be ordered, but that any polynomial equation with real coefficients can be solved in the complex field
d. Identify and translate between equivalent forms of algebraic expressions and equations using a variety of techniques (e.g., factoring, applying properties of operations)
e. Justify the steps in manipulating algebraic expressions and solving algebraic equations and inequalities
f. Represent situations and solve problems using algebraic equations and inequalities

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: The Real Number System, High School [N-RN]; The Complex Number System, High School [N-CN]; Seeing Structure in Expressions, High School [A-SSE]; Reasoning with Equations and Inequalities, High School [A-REI]; Creating Equations, High School [A-CED])

1.2 Polynomial Equations and Inequalities

a. Analyze and solve polynomial equations with real coefficients using:
   - the Fundamental Theorem of Algebra
   - the Rational Root Theorem for polynomials with integer coefficients
   - the Conjugate Root Theorem for polynomial equations with real coefficients
   - the Binomial Theorem
b. Prove and use the Factor Theorem and the quadratic formula for real and complex quadratic polynomials
c. Solve polynomial inequalities

(California Common Core Content Standards for Mathematics, including Standards for

Mathematical Practice 1–8: Reasoning with Equations and Inequalities, High School [A-REI]; Arithmetic with Polynomials and Rational Expressions, High School [A-APR]; Linear, Quadratic, and Exponential Models, High School [F-LE])

1.3 Functions

a. Analyze general properties of functions (i.e., domain and range, one-to-one, onto, inverses, composition, and differences between relations and functions) and apply arithmetic operations on functions
b. Analyze properties of linear functions (e.g., slope, intercepts) using a variety of representations
c. Demonstrate knowledge of why graphs of linear inequalities are half planes and be able to apply this fact
d. Analyze properties of polynomial, rational, radical, and absolute value functions in a variety of ways (e.g., graphing, solving problems)
e. Analyze properties of exponential and logarithmic functions in a variety of ways (e.g., graphing, solving problems)
f. Model and solve problems using nonlinear functions

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Interpreting Functions, High School [F-IF]; Building Functions, High School [F-BF]; Linear, Quadratic, and Exponential Models, High School [F-LE])

1.4 Linear Algebra

a. Understand and apply the geometric interpretation and basic operations of vectors in two and three dimensions, including their scalar multiples
b. Prove the basic properties of vectors (e.g., perpendicular vectors have zero dot product)
c. Understand and apply the basic properties and operations of matrices and determinants (e.g., to determine the solvability of linear systems of equations)
d. Analyze the properties of proportional relationships, lines, linear equations, and their graphs, and the connections between them
e. Model and solve problems using linear equations, pairs of simultaneous linear equations, and their graphs

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Vector and Matrix Quantities, High School [N-VM]; Expressions and Equations, Grade 8; Linear, Quadratic, and Exponential Models, High School [F-LE]; Ratios and Proportional Relationships, Grade 7 [7.RP])

Domain 2: Geometry

Candidates demonstrate an understanding of the foundations of geometry as outlined in the California Common Core Content Standards for Mathematics (Grade 7, Grade 8, and High School). Candidates demonstrate a depth and breadth of conceptual knowledge to ensure a

rigorous view of geometry and its underlying structures. They demonstrate an understanding of axiomatic systems and different forms of logical arguments. Candidates understand, apply, and prove theorems relating to a variety of topics in two- and three-dimensional geometry, including coordinate, synthetic, non-Euclidean, and transformational geometry.

2.1 Plane Euclidean Geometry

a. Apply the Parallel Postulate and its implications and justify its equivalents (e.g., the Alternate Interior Angle Theorem, the angle sum of every triangle is 180 degrees)
b. Demonstrate knowledge of complementary, supplementary, and vertical angles
c. Prove theorems, justify steps, and solve problems involving similarity and congruence
d. Apply and justify properties of triangles (e.g., the Exterior Angle Theorem, concurrence theorems, trigonometric ratios, triangle inequality, Law of Sines, Law of Cosines, the Pythagorean Theorem and its converse)
e. Apply and justify properties of polygons and circles from an advanced standpoint (e.g., derive the area formulas for regular polygons and circles from the area of a triangle)
f. Identify and justify the classical constructions (e.g., angle bisector, perpendicular bisector, replicating shapes, regular polygons with 3, 4, 5, 6, and 8 sides)

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Geometry, Grade 7 [7.G]; Geometry, Grade 8; Congruence, High School [G-CO]; Similarity, Right Triangles, and Trigonometry, High School [G- SRT]; Circles, High School [G-C]; Geometric Measurement and Dimension, High School [G-GMD])

2.2 Coordinate Geometry

a. Use techniques in coordinate geometry to prove geometric theorems
b. Model and solve mathematical and real-world problems by applying geometric concepts to two-dimensional figures
c. Translate between the geometric description and the equation for a conic section
d. Translate between rectangular and polar coordinates and apply polar coordinates and vectors in the plane

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Geometry, Grade 8; Expressing Geometric Properties with Equations, High School [G-GPE]; Geometric Measurement and Dimension, High School [G-GMD]; Modeling with Geometry, High School [G-MG]; Polar Coordinates and Curves, High School)

2.3 Three-Dimensional Geometry

a. Demonstrate knowledge of the relationships between lines and planes in three dimensions (e.g., parallel, perpendicular, skew, coplanar lines)

b. Apply and justify properties of three-dimensional objects (e.g., the volume and surface area formulas for prisms, pyramids, cones, cylinders, spheres)
c. Model and solve mathematical and real-world problems by applying geometric concepts to three-dimensional figures

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Congruence, High School [G-CO]; Similarity, Right Triangles, and Trigonometry, High School [G-SRT]; Geometric Measurement and Dimension, High School [G-GMD]; Modeling with Geometry, High School [G-MG])

2.4 Transformational Geometry

a. Demonstrate knowledge of isometries in two- and three-dimensional space (e.g., rotation, translation, reflection), including their basic properties in relation to congruence
b. Demonstrate knowledge of dilations (e.g., similarity transformations or change in scale factor), including their basic properties in relation to similarity, volume, and area

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Geometry, Grade 8; Congruence, High School [G-CO])

Domain 3: Number and Quantity

Candidates demonstrate an understanding of number theory and a command of number sense as outlined in the California Common Core Content Standards for Mathematics (Grade 6, Grade 7, Grade 8, and High School). Candidates demonstrate a depth and breadth of conceptual knowledge to ensure a rigorous view of number theory and its underlying structures. They prove and use properties of natural numbers. They formulate conjectures about the natural numbers using inductive reasoning, and verify conjectures with proofs.

3.1 The Real and Complex Number Systems

a. Demonstrate knowledge of the properties of the real number system and of its subsets
b. Perform operations and recognize equivalent expressions using various representations of real numbers (e.g., fractions, decimals, exponents)
c. Solve real-world and mathematical problems using numerical and algebraic expressions and equations
d. Apply proportional relationships to model and solve real-world and mathematical problems
e. Reason quantitatively and use units to solve problems (i.e., dimensional analysis)
f. Perform operations on complex numbers and represent complex numbers and their operations on the complex plane

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: The Number System, Grade 7 [7.NS]; The Real Number

System, Grade 8; Quantities, High School [N-Q]; Expressions and Equations, Grade 7 [7.EE]; Ratios and Proportional Relationships, Grade 7 [7.RP]; The Real Number System, High School [N-RN]; The Complex Number System, High School [N-CN])

3.2 Number Theory

a. Prove and use basic properties of natural numbers (e.g., properties of divisibility)
b. Use the principle of mathematical induction to prove results in number theory
c. Apply the Euclidean Algorithm
d. Apply the Fundamental Theorem of Arithmetic (e.g., find the greatest common factor and the least common multiple; show that every fraction is equivalent to a unique fraction where the numerator and denominator are relatively prime; prove that the square root of any number, not a perfect square number, is irrational)

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: The Number System, Grade 6 [6.NS])

Domain 4: Probability and Statistics

Candidates demonstrate an understanding of statistics and probability distributions as outlined in the California Common Core Content Standards for Mathematics (Grade 7, Grade 8, and High School). Candidates demonstrate a depth and breadth of conceptual knowledge to ensure a rigorous view of probability and statistics and their underlying structures. They solve problems and make inferences using statistics and probability distributions.

4.1 Probability

a. Prove and apply basic principles of permutations and combinations
b. Illustrate finite probability using a variety of examples and models (e.g., the fundamental counting principles, sample space)
c. Use and explain the concepts of conditional probability and independence
d. Compute and interpret the probability of an outcome, including the probabilities of compound events in a uniform probability model
e. Use normal, binomial, and exponential distributions to solve and interpret probability problems
f. Calculate expected values and use them to solve problems and evaluate outcomes of decisions

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Statistics and Probability, Grade 7 [7.SP]; Conditional Probability and the Rules of Probability, High School [S-CP]; Using Probability to Make Decisions, High School [S-MD])

4.2 Statistics

a. Compute and interpret the mean and median of both discrete and continuous distributions
b. Compute and interpret quartiles, range, interquartile range, and standard

deviation of both discrete and continuous distributions

c. Select and evaluate sampling methods appropriate to a task (e.g., random, systematic, cluster, convenience sampling) and display the results
d. Apply the method of least squares to linear regression
e. Apply the chi-square test
f. Interpret scatter plots for bivariate data to investigate patterns of association between two quantities (e.g., correlation), including the use of linear models
g. Interpret data on a single count or measurement variable presented in a variety of formats (e.g., dot plots, histograms, box plots)
h. Demonstrate knowledge of P-values and hypothesis testing
i. Demonstrate knowledge of confidence intervals

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Statistics and Probability, Grade 8; Interpreting Categorical and Quantitative Data, High School [S-ID])

Domain 5: Calculus

Candidates demonstrate an understanding of trigonometry and calculus as outlined in the California Common Core Content Standards for Mathematics (High School). Candidates demonstrate a depth and breadth of conceptual knowledge to ensure a rigorous view of trigonometry and calculus and their underlying structures. They apply the concepts of trigonometry and calculus to solving problems in real-world situations.

5.1 Trigonometry

a. Prove that the Pythagorean Theorem is equivalent to the trigonometric identity $\sin^2 x + \cos^2 x = 1$ and that this identity leads to $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$
b. Prove and apply the sine, cosine, and tangent sum formulas for all real values
c. Analyze properties of trigonometric functions in a variety of ways (e.g., graphing and solving problems, using the unit circle)
d. Apply the definitions and properties of inverse trigonometric functions (i.e., arcsin, arccos, and arctan)
e. Apply polar representations of complex numbers (e.g., DeMoivre's Theorem)
f. Model periodic phenomena with periodic functions
g. Recognize equivalent identities, including applications of the half-angle and double- angle formulas for sines and cosines

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Trigonometric Functions, High School [F-TF])

5.2 Limits and Continuity

a. Derive basic properties of limits and continuity, including the Sum, Difference,

Product, Constant Multiple, and Quotient Rules, using the formal definition of a limit

b. Show that a polynomial function is continuous at a point
c. Apply the intermediate value theorem, using the geometric implications of continuity

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Calculus Standards, High School)

5.3 Derivatives and Applications

a. Derive the rules of differentiation for polynomial, trigonometric, and logarithmic functions using the formal definition of derivative
b. Interpret the concept of derivative geometrically, numerically, and analytically (i.e., slope of the tangent, limit of difference quotients, extrema, Newton's method, and instantaneous rate of change)
c. Interpret both continuous and differentiable functions geometrically and analytically and apply Rolle's theorem, the mean value theorem, and L'Hôpital's rule
d. Use the derivative to solve rectilinear motion, related rate, and optimization problems
e. Use the derivative to analyze functions and planar curves (e.g., maxima, minima, inflection points, concavity)
f. Solve separable first-order differential equations and apply them to growth and decay problems

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Calculus Standards, High School)

5.4 Integrals and Applications

a. Derive definite integrals of standard algebraic functions using the formal definition of integral
b. Interpret the concept of a definite integral geometrically, numerically, and analytically (e.g., limit of Riemann sums)
c. Prove the fundamental theorem of calculus, and use it to interpret definite integrals as antiderivatives
d. Apply the concept of integrals to compute the length of curves and the areas and volumes of geometric figures

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Calculus Standards, High School)

5.5 Sequences and Series

a. Derive and apply the formulas for the sums of finite arithmetic series and finite and infinite geometric series (e.g., express repeating decimals as a rational

number)

b. Determine convergence of a given sequence or series using standard techniques (e.g., ratio, comparison, integral tests)
c. Calculate Taylor series and Taylor polynomials of basic functions

(California Common Core Content Standards for Mathematics, including Standards for Mathematical Practice 1–8: Seeing Structure in Expressions, High School [A-SSE]; Calculus Standards, High School)

E. Review and Approval of Subject Matter Programs

A regionally accredited institution of postsecondary education that would like to offer a subject matter preparation program may submit a program proposal that responds to the standards and preconditions in this handbook. The submission of programs for review and approval is voluntary for colleges and universities, but only Commission-approved subject matter programs will meet a prospective teacher's requirement for subject matter competence. Submissions are reviewed by a program review panel with guidance from Commission staff. Institutions are encouraged to support faculty and/or instructional staff with subject matter expertise in participating as members of program review panels.

Selection and Composition of Program Review Panels
Review panel members are selected based on their expertise in a particular subject area. Panel members are also expected to be knowledgeable about K-12 curriculum and instruction in California public schools. Reviewers are selected from institutions of higher education, school districts, county offices of education, organizations of subject matter experts, and statewide professional organizations.

Steps in the Review of Programs
The Commission is committed to conducting a program review process that is objective, authoritative, and comprehensive. The Commission also seeks to be as helpful as possible to colleges and universities throughout the review process. Commission staff is available to guide and assist programs throughout the process.

Review of Preconditions: Commission staff typically review preconditions because preconditions address issues of compliance and not program quality. This review is done prior to initial program approval. Once the Preconditions have been determined to have been met, the program document is referred to the expert review panel for review against the program standards.

Review of Program Standards: Since the program standards address issues of program quality and effectiveness, each institution's response to the standards is reviewed by a review panel of subject matter experts. If the panel determines that a proposed program meets the standards, Commission staff will recommend the program for approval by the Committee on Accreditation.

If the program review panel determines that the program does not meet the standards, the document will be returned to the institution with an explanation of the panel's findings, including specific reasons for the findings. If the panel has substantive concerns about one or more aspects of program quality, representatives of the institution can obtain further information and technical assistance from Commission staff so that the program review process is as helpful as possible to colleges and universities.

If the Program Review Panel determines that minor or technical changes should be made by the program sponsor in order to meet the program standards, Commission staff will review the resubmitted document and recommend the program for approval by the Committee on Accreditation.

There is no penalty for programs that do not meet all subject matter program standards on the first try. However, an institution may not offer a program to prospective candidates until the program is approved by the Commission.

Appeal of an Adverse Decision: An institution that would like to appeal a decision of the staff regarding the preconditions review or of the program review panel regarding the standards review, it may do so by submitting the appeal to the Commission's Executive Director. The institution should include the following information in the appeal:

- A rationale for the appeal by the institution
- The original program document and the stated reasons of the Commission's staff and/or the review panel for not recommending approval of the program.
- A specific response by the institution to the initial denial, including a copy of the resubmitted document (if it has been resubmitted).

The Executive Director may deny the appeal or appoint an independent review panel or present the appeal directly to the Commission for consideration.